ALGORITHM FOR SOLVING PLASTICITY PROBLEMS BY THE FINITE-ELEMENT METHOD

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An algorithm developed in [1] for solving plasticity problems by calculating stresses for different time intervals is used to exactly determine the moments at which the yield point is attained and plastic flow begins. A problem concerning the tension of a plate with semi-circular notches is solved to illustrate the use of the algorithm.

The equations of the theory of plastic flow with linear strain-hardening have the form [1, 2]

$$\begin{gathered}
\dot{\sigma}_{ij}' = 2G(\dot{e}_{ij}' - \dot{\eta}_{ij}), \quad \dot{\sigma}_{nn} = 3K\dot{e}_{nn}, \\
\dot{s}_{ij} = \theta\dot{\eta}_{ij}, \quad \dot{\eta}_{ij} = 3\Lambda(\sigma_{ij}' - s_{ij}), \quad s_{nn} = 0; \\
\sigma_{ij}' = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{nn}, \quad e_{ij}' = e_{ij} - \frac{1}{3}\delta_{ij}e_{nn} \quad (i, j = 1, 2, 3), \\
\Lambda = c\chi W, \quad \chi = \frac{G}{(2G + \theta_1)S^2}, \\
W = (\sigma_{ij}' - s_{ij})\dot{e}_{ij}', \quad S^2 = \frac{3}{2}(\sigma_{ij}' - s_{ij})(\sigma_{ij}' - s_{ij}), \\
c = \begin{cases} 0, & \text{if} \quad S < S_* \text{ or } S = S_*, \ W \le 0, \\ 1, & \text{if} \quad S = S_*, \ W > 0, \end{cases} \\
G = \frac{E}{2(1+\nu)}, \quad K = \frac{E}{3(1-2\nu)}, \quad \theta_1 = \frac{2}{3}\cdot\frac{EE_1}{E - E_1}.
\end{gathered} \tag{1}$$

Here, E and ν are the elastic modulus and Poisson's ratio; E_1 is the shear modulus on the plastic section of the strain dependence of stress in the uniaxial stress state; E_1, θ_1, and θ are material constants ($0 \le \theta \le \theta_1$; in particular, $\theta = 0$, $\theta_1 > 0$ for purely isotropic strain-hardening, $\theta = \theta_1 > 0$ for purely translational strain-hardening, and $\theta = \theta_1 = 0$ for ideal plasticity); S is a function of the loading surface; S_* is the yield point; e_{ij}, η_{ij}, and σ_{ij} are components of the tensor of the total and plastic strains and stresses, respectively; s_{ij} are the coordinates of the center of the loading surface; δ_{ij} is the Kronecker symbol; i, j = 1, 2, 3; fulfilling the summation by reiterative indices i, j, n = 1, 2, 3; the primes denote a change to components of the tensor of the deviator; a superimposed dot denotes differentiation with respect to the loading parameter (time) τ.

We will describe the calculation of the stresses for the interval from τ to $\tau + \Delta\tau$ at each node of a formula for integration over the volume of an element (or at each point of integration). Using the algorithm in [1], we have the strains $e_{ij\tau}$, $e_{ij\tau+\Delta\tau}$ at the beginning and end of the time step (here and below, values of the function will be denoted by the subscript corresponding to the moment to which they pertain). We assume that the strain rates are constant over the entire interval and are equal to $e_{ij}^* = (e_{ij\tau+\Delta\tau} - e_{ij\tau})/\Delta\tau$ (i, j = 1, 2, 3).

In the case of purely elastic deformation, the stresses have the values

$$\sigma_{ijt} = \sigma_{ij\tau} + (t - \tau)\,2G\left(\dot{e}_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\dot{e}_{nn}\right) \quad (t \geqslant \tau) \quad (i, j = 1, 2, 3). \tag{2}$$

We find the points of intersection of ray (2) with the loading surface either as the root t of the equation $S_t^2 = S_{*\tau}^2$ or as

$$a_0(t - \tau)^2 + 2a_1(t - \tau) + a_2 = 0, \tag{3}$$

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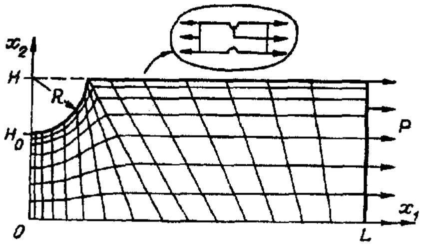

Fig. 1

where

$$a_0 = 6G^2 \dot{e}_{ij}'\dot{e}_{ij}'; \quad a_1 = 3GW_\tau; \quad a_2 = S_\tau^2 - S_{*\tau}^2.$$

Purely elastic deformation is assumed to occur in the interval from τ to the moment τ_1 – the largest root $\tau_1 = t$ in (3). Here, $\tau_1 > \tau$. If there are no real roots and $S_\tau > S_{*\tau}$, $W_\tau < 0$, we assign $\tau_1 = \tau - a_1/a_0$ and, using (2) from τ to τ_1, we reduce the value of the function S. In the case $\tau_1 > \tau + \Delta\tau$, we use Eqs. (2) to the end of the interval.

On the remaining part of the step from τ_1 to $\tau + \Delta\tau$, where plastic flow occurs, we assume

$$\begin{gathered} \sigma_{ij\tau_2} = \sigma_{ij\tau_1} + (\tau_2 - \tau_1)\,\dot{\sigma}_{ij\tau_2}, \\ s_{ij\tau_2} = s_{ij\tau_1} + (\tau_2 - \tau_1)\,\dot{s}_{ij\tau_2} \quad (i,\, j = 1,\, 2,\, 3), \\ \tau_2 = \frac{1}{2}(\tau + \Delta\tau + \tau_1). \end{gathered} \tag{4}$$

Inserting the expressions for $\dot{\sigma}_{ij}$ and $\dot{s}_{ij}$ from (1) into (4) with $c = 1$, we solve the resulting equations relative to $\sigma_{ij\tau 2}$, $s_{ij\tau 2}$ by iteration [1]. We then calculate

$$\sigma_{ij\tau+\Delta\tau} = 2\sigma_{ij\tau_2} - \sigma_{ij\tau_1}, \quad s_{ij\tau+\Delta\tau} = 2s_{ij\tau_2} - s_{ij\tau_1} \quad (i,\, j = 1,\, 2,\, 3). \tag{5}$$

It follows from (4) and (5) that

$$\begin{gathered} W_{\tau+\Delta\tau} = W_{\tau_1} + (\tau + \Delta\tau - \tau_1)\,\dot{W}_{\tau_2}, \quad W_{\tau_2} = \frac{1}{2}(W_{\tau+\Delta\tau} + W_{\tau_1}), \\ S^2_{\tau+\Delta\tau} = S^2_{\tau_1} + (\tau + \Delta\tau - \tau_1)\,(S^2)^{\cdot}_{\tau_2}. \end{gathered}$$

Here, we take the values of the functions

$$\begin{gathered} \dot{W} = 2G\left(\dot{e}_{ij}'\dot{e}_{ij}' - \frac{2G + \theta}{2G + \theta_1}\,\frac{3W^2}{2S^2}\right), \\ (S^2)^{\cdot} = \frac{6G(\theta_1 - \theta)}{2G + \theta_1}\,W. \end{gathered}$$

We change over to the sequence of inequalities

$$\begin{gathered} \theta \leqslant \theta_1, \quad \dot{e}_{ij}'\dot{e}_{ij}' \geqslant \frac{3W^2}{2S^2}, \quad \dot{W} \geqslant 0, \quad W_{\tau+\Delta\tau} \geqslant W_{\tau_1} \geqslant 0, \\ W_{\tau_2} \geqslant 0, \quad (S^2)^{\cdot}_{\tau_2} \geqslant 0, \quad S_{\tau+\Delta\tau} \geqslant S_{\tau_1} \geqslant S_{*\tau}, \end{gathered}$$

while $S_{\tau+\Delta\tau} = S_{\tau 1}$ in the cases of ideal plasticity or purely translational strain-hardening. We have thus proven that the conditions for plastic flow at the end of the interval $S_{\tau+\Delta\tau} = S_{*\tau+\Delta\tau} \geq S_{*\tau}$, $W_{\tau+\Delta\tau} \geq 0$ are not violated. The step is divided into two subintervals: before τ_1, we have purely elastic deformation; from τ_1 to the end of the step, we have elastoplastic deformation.

The increments of stress and plastic strain are determined more accurately than on the step as a whole [1]. Errors in the satisfaction of the condition $S \leq S_*$ are eliminated, making it possible to perform a calculation with larger time steps. The moments at which plastic flow begins with a step are determined in accordance with the algorithm proposed in [3].

Let us examine the solution of a problem concerning the equilibrium of a plate with semi-circular notches. The plate is subjected to tension by a load P which is distributed uniformly on its edges and increases monotonically (Fig. 1). We will use Eqs. (1) for a material which is elastic and ideally plastic. We will also make use of linear expressions for the strains

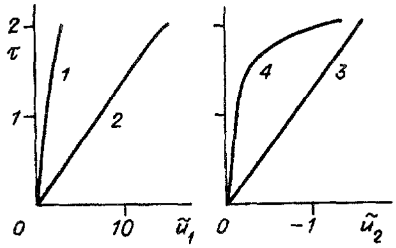

Fig. 2

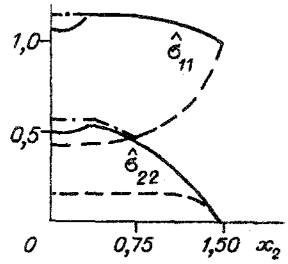

Fig. 3

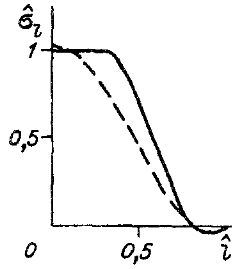

Fig. 4

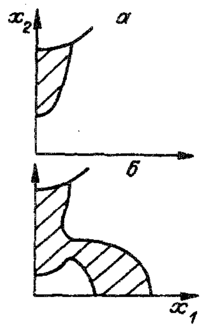

Fig. 5

(written in displacements) and equilibrium equations [2]. The problem will be solved in the Cartesian coordinates x_1, x_2. The case of a plane strain state will be considered.

Due to the symmetry of the problem, we will obtain the solution for one-fourth of the plate (Fig. 1) (x_1 and x_2 are the axes of symmetry, R is the radius of the notch, L = 6R, H = 2.5R, and H_0 = 1.5R). The edge x_2 = H, and the contour of the notch is not loaded. On the edge x_1 = L, we assign $\sigma_{11} = P$, $\sigma_{12} = 0$. The given one-fourth of the plate is broken up into 105 tetragonal isoparametric Lagrangian finite elements with nine nodes (Fig. 1). We have 884 sought variables – the components of the displacements – at the nodes. We will use the three-point Gaussian formula [3] to calculate the integrals over the area of each element.

We introduce dimensionless quantities (denoted by a superimposed ~):

$$\tilde{x}_i = \frac{1}{R} x_i, \quad \tilde{u}_i = \frac{Eu_i}{(1 - \nu^2) RP_0}, \quad \tilde{\sigma}_{ij} = \frac{1}{P_0} \sigma_{ij} \quad (i, j = 1, 2),$$
$$\tilde{S} = \frac{1}{P_0} S, \quad \tilde{S}_* = \frac{1}{P_0} S_*, \quad \tau = \frac{P}{P_0}$$

(u_1 and u_2 are the displacements in the direction of the x_1 and x_2 axes; P_0 is the load at which the yield point is reached in the elastic plate). The Poisson's ratio ν = 0.3. In the given dimensionless variables, the solution is independent of E, R, and P_0.

At $\tau = 1$, maximum stress intensity at the points of integration $\tilde{S} = \tilde{S}_* = 3.0407$ is assumed to be equal to the yield point. Interpolating the stresses from the points of integration to the nodes of the elements, we determine the stress-intensity factor on the edge of the notches – $\tilde{\sigma}_{11} = 3.1123$ at the point (0, H_0) (for an infinitely long plate ($L \to \infty$), the stress-concentration factor is equal to 3.109 [4, 5]). Equations (1) are satisfied only at the points of integration, so that stress intensity at the other points can – as the given value of $\tilde{\sigma}_{11}$ – be only slightly greater than the yield point.

The calculation for plastic strains is performed in the load interval $1 \le \tau \le T = 2.045$ on a sequence of 22 steps: four steps with $\Delta\tau = 0.075$; 11 steps with $\Delta\tau = 0.05$; 6 steps with $\Delta\tau = 0.03$; 1 step with $\Delta\tau = 0.015$. The given value of T is less than the load $T_H = 2.0567$ obtained in [6, 2] for a plastic-rigid plate. It should be noted that the necessary number of iterations of displacement rate per step increases (to 15 for the last step) with an increase in τ, as does the computing time for each interval.

At $\tau > 1$, the displacements in the region outside certain neighborhoods of the notches continue to increase almost linearly in relation to τ. Lines 1-4 in Fig. 2 show the effect of the value of τ on $\tilde{u}_1$ at the points (R, 0), (L, 0) and $\tilde{u}_2$ at the points (L, H), (0, H_0), respectively. The rates $\dot{u}_2$ (under the notch) and $\dot{u}_1$ increase sharply as τ approaches T.

Figure 3 shows the distributions of the value $\hat{\sigma}_{11} = \sigma_{11}/S_*$, $\hat{\sigma}_{22} = \sigma_{22}/S_*$ in the section $x_1 = 0$, while Fig. 4 shows the distribution of the hoop stress σ_l on the edge of a notch over the length of an arc l reckoned from the point $(0, H_0)$, with $\hat{\sigma}_l = \sigma_l/S_*$, $\hat{l} = 2l/(\pi R)$. The dashed lines in Fig. 3 pertain to $\tau = 1$, the solid lines to $\tau = T$, and the dot-dash lines to the values at which these lines diverge from the solid lines – the values of $\hat{\sigma}_{11}$, $\hat{\sigma}_{22}$ in the solution [6, 2].

The hatched regions in Fig. 5 show the regions of plastic flow in the central part of the plate under the notch for $\tau = 2$ and T (a and b). The boundaries of these regions have been drawn approximately as smooth curves enveloping discrete sets of points of integration at which the rates of plastic strain are nontrivial. At $\tau = T$, the plastic flow regions, positioned symmetrically in each quadrant of the coordinate system x_1, x_2, merge on the x_1 axis. In the given solution, as τ increases the equilibrium of the plate is assured by expansion of the region of plastic flow and stress distribution the displacements and strains remain finite. Nevertheless, on the basis of safety considerations, T should be considered the maximum allowable load $T = T_*$. Of course, the value of T_* can be refined by making the steps $\Delta\tau$ smaller. An additional indication of the attainment of T_* is a sharp increase in displacement rate.

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